ESTIMATING COMPLEXITY OF 2D SHAPES

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ABSTRACT

This paper deals with the problem of estimating 2D shape complexity. This has important applications in computer vision as well as in developing efficient shape classification algorithms. We define shape complexity using correlates of Kolmogorov complexity – entropy measures of global distance and local angle, and a measure of shape randomness. We tested our algorithm on synthetic and real world datasets with excellent results. We also conducted user studies that indicate that our measure is highly correlated with human perception. They also reveal an intuitive shape sensitivity curve – simple shapes are easily distinguished by small complexity variations, while complex shapes require significant complexity differences to be differentiated.

1. INTRODUCTION

In this paper, we define 2D shape complexity and provide an algorithm to estimate shape complexity. The problem is important in areas such as computer vision [6], geographic information systems [1] and neural structure analysis. Shape complexity plays an important role in computationally efficient shape classification algorithms [3]. Intuitively, if small differences amongst complex shapes cannot be discerned, full shape analysis is not required for classification. Figure 1 shows a typical 2D shape classification problem, where the shape contour is derived from the video.

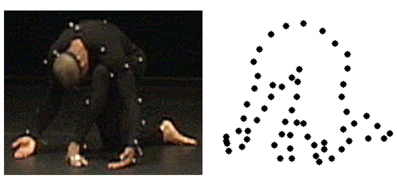
(a) (b)

Figure 2: (a) dancer's pose, (b) contour

There has been prior work on shape complexity. Toussaint [7] proposed a measure for shape complexity based on polygon triangulation. In [2], a measure of complexity based on sinuosity is presented. In [5], a shape complexity based on the entropy of curvature of object contour is presented. Prior work does not use perceptual factors in estimating shape complexity. Importantly, past work in shape similarly does not take shape complexity into account [8].

We formally define shape complexity using Kolmogorov complexity. We estimate the shape complexity using strong correlates of Kolmogorov complexity – entropy measures and a measure of randomness. We first compute the global distance entropy and local angle entropy at different resolutions. Then we select an optimal resolution to trade-off entropy with quantization error. Then, we estimate *shape randomness* using a novel shape trace difference metric. Finally we combine these parameters to compute the shape complexity. We tested our algorithm on a large synthetic and real-world dataset with excellent results. We also conducted user studies to determine human validation of our complexity measure. The studies indicate that the measure is well correlated with perception and also reveal the existence of a shape complexity-difference sensitivity curve.

2. SHAPE COMPLEXITY DEFINITION

We formally define the shape complexity to be a function of its Kolmogorov complexity [4]. Let *s* be a shape with *n* points. Let *U(p)* denote the output of a universal Turing machine *U* when input with program *p*. Then the shape complexity *C(s)* of a 2D shape *s* is defined as:

$$C(s) \triangleq K_U(s \mid n) = \min_{p:U(p)=s} l(p), \qquad \langle 1 \rangle$$

where $K_U(s|n)$ is Kolmogorov complexity of a shape *s* with *n* points and *l(p)* is the length of program *p*. This complexity definition captures only the structural information of the shape. It does not capture how the shape is perceived. The perceptual factors are critical in differentiating between two shapes [8], and we conjecture that the difference between two shapes is a function of the structural complexity and the perception of the two shapes. In this paper, we shall estimate the structural shape complexity and derive its relationship to perception with user studies.

The Kolmogorov complexity of a number is non-computable [4]. Instead we shall use strong correlates of Kolmogorov complexity – measures of randomness and entropy, to derive our complexity measure. The structural shape complexity will be estimated using entropy of the global distance distribution (GDD), entropy of the local angle distribution (LAD) and a randomness measure. The GDD is the distribution of distances of all points in the shape to their centroid. If the distances from points of a shape to their centroid fall into a small range, then the shape is a simple circle or circle-like shape (see Figure 3 a, b) whose complexity should be small. But for some shapes (Figure 3 c, d) whose distance distributions are more widespread also look simple since their contours are very smooth. Thus, we use LAD to measure shape's local smoothness and regularity. The last consideration is randomness. This is based on the intuition that it is easy to draw a unique contour for simple shapes (Figure 3 a-d) because of the high correlation between points, whereas it is hard to draw a unique contour for a random shape (Figure 3 e).

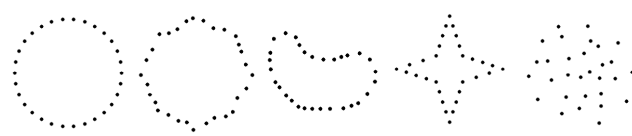
(a) (b) (c) (d) (e)

Figure 3 Five examples of 2D shape

3. GLOBAL DISTANCE ENTROPY

In this section, we shall discuss a multi-resolution global distance entropy definition and optimal resolution selection.

3.1 Normalized Global Distance

Let us denote the unlabeled 2D point coordinates of a shape as $\zeta=\{X_i=(x_i,y_i)^T,\ i=1,\ldots,N\}$ where *N* is the number of points. We create an object centric coordinate system, by moving the origin to the centroid of *N* markers. We extract the distance from each

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marker to the center denoted as r_i and divide r_i by the maximum distance to normalize them to interval [0,1] (see Figure 4 (a)).

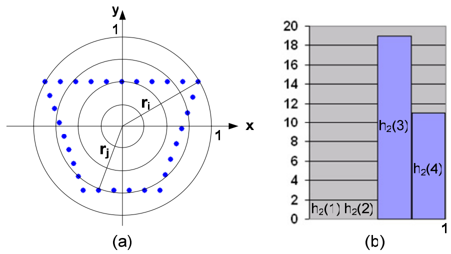


Figure 4: (a) shape in normalized 2D space that is divided into four distance bins. The shape centroid is the origin, r_i and r_j are normalized distances of i^{th} and j^{th} markers. (b) Distance histogram**.**

3.2 Distance Histogram and Entropy

The distance values r_i of the 2D points form an unordered set. This is because the markers do not have any labels associated with them. As the first step towards 2D shape representation, we transform the distance values into a distance histogram. At resolution J, we uniformly divide the normalized distance space [0,1] into $K=2^J$ bins. Thus, the histogram with J resolutions – $h_J(k)$ is represented as:

$$h_J(k) = |\{r_i | r_i \in [\frac{k-1}{2^J}, \frac{k}{2^J}), i = 1,..,N\}|, 1 \le k \le 2^J \quad ,\qquad \text{<2>}$$

where N is the number of points and |·| is cardinality (set size) operator. Figure 4 (a) shows the case where the normalized 2D space is divided into 4 distance bins and Figure 4 (b) shows the distance histogram based on this division where $J=2$, $h_2(1)$, $h_2(2)$, $h_2(3)$ and $h_2(4)$ are the number of points in the four bins.

With distance histogram, we can easily estimate the pdf of distance distribution $p_J(k)$ at resolution J by dividing $h_J(k)$ by N. Thus, we can define the distance entropy at resolution J as:

$$H_{J,dist} = -\sum_{k=1}^{2^J} p_J(k) \cdot \log_2[p_J(k)] \qquad \text{<3>}$$

where $H_{J,dist}$ is distance entropy at resolution J.

3.3 Resolution Selection

In this section, we shall show how to select optimal resolution. In Eq.<3>, it is observed that distance entropy is a function of resolution J or number of bins. We need to select an optimal resolution J^* and compute the distance entropy at this resolution. Our selection algorithm is based on distance entropy (eq.<3>) and quantization error. We first discuss quantization error and then define a cost function to select resolution.

Let us denote $Q_J[\cdot]$ as quantization operator and $V_{k,J}=[(k-1)/2^J, k/2^J)$ $1 \le k \le 2^J$ as the kth quantization level at resolution J. Thus, given a distance r_i, we have:

$$Q_J[r_i] = k, if \quad r_i \in V_{k,J} \qquad \text{<4>}$$

Hence, given a distance set of a shape $R=\{r_i\}$, we compute quantization values $\mu_{k,J}$ at resolution J by:

$$\mu_{k,J} = \frac{1}{|\xi_k|} \sum_{r_i \in \xi_k} r_i, \; \xi_k = \{r_i | r_i \in V_{k,J}\}, 1 \le k \le 2^J \qquad \text{<5>}$$

where ξ_k is a subset of R in which each component falls into $V_{k,J}$. Based on quantization level $V_{k,J}$ and quantization value $\mu_{k,J}$, the quantization error at resolution J is represented as follows:

$$e_{J,dist} = \sqrt{\frac{1}{N} \cdot \sum_{i=1}^{N} (r_i - \mu_{Q_J[r_i],J})^2}, \qquad \text{<6>}$$

As resolution J increases, the distance entropy will increase while quantization error will decrease since the space is divided more finely. Thus we define a cost function to trade-off entropy with quantization error:

$$f_{dist}(J) = \frac{H_{J,dist}}{\log_2 N} + \frac{e_{J,dist}}{e_{\max}}, \qquad \text{<7>}$$

where $H_{J,dist}$ and $e_{J,dist}$ are distance entropy and quantization error at resolution J respectively. The cost function is then the sum of normalized entropy and normalized quantization error in which log_2N is the maximum distance entropy for a shape with N points and e_{max} is a constant which represents the maximum possible quantization error over all resolutions. Using cost function Eq.<7>, we select the resolution with minimum cost as the optimum resolution J^*, and define the cost value at resolution J^* as the *global distance factor* C^e_{dist} of shape complexity:

$$C^e_{dist} = \min_j [f_{dist}(j)]. \qquad \text{<8>}$$

4. LOCAL ANGLE ENTROPY

4.1 Local Angle

For each point $X_i=(x_i,y_i)^T$, we extract a local angle based on neighbor points by using following procedures (also ref. Figure 5). This is important for general point sets, when the shape contour is not known.

1. Search for the nearest neighbor of X_i, denote it as $X_{i,nn1}$.
2. Compute the prediction point based on X_i and $X_{i,nn1}$ by:

$$X_{i,p} = \begin{bmatrix} x_{i,p} \\ y_{i,p} \end{bmatrix} = \begin{bmatrix} x_i + (x_i - x_{i,nn1}) \\ y_i + (y_i - y_{i,nn1}) \end{bmatrix}, \qquad \text{<9>}$$

3. Search for the nearest neighbor of $X_{i,p}$ from point set $\zeta \backslash \{X_i, X_{i,nn1}\}$ where $\zeta=\{X_i=(x_i,y_i)^T, i=1,\ldots,N\}$ and denote it as $X_{i,nn2}$. Hence the local angle of point X_i is defined as:

$$\theta_i = \begin{cases} \angle X_{i,nn1} X_i X_{i,nn2} & if \quad \angle X_{i,nn1} X_i X_{i,nn2} \le \pi \\ 2\pi - \angle X_{i,nn1} X_i X_{i,nn2} & otherwise \end{cases} \qquad \text{<10>}$$

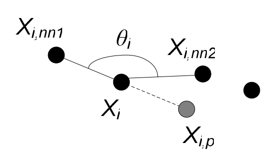


Figure 5: Local angle computation.

4.2 Angle Histogram and Entropy

We represent the local angle information of a shape using angle histogram and represent angle complexity using angle entropy. At resolution J, we uniformly divide the angle space [0, π] into 2^J bins and represent angle histogram $\eta_J(k)$ as:

$$\eta_J(k) = |\{\theta_i | \theta_i \in [\frac{(k-1)\pi}{2^J}, \frac{k\pi}{2^J}), i = 1,..,N\}|, 1 \le k \le 2^J \qquad \text{<11>}$$

Similar with section 3.2, we estimate the pdf of angle distribution and define the angle entropy at resolution J.

$$H_{J,angle} = -\sum_{k=1}^{2^J} q_J(k) \cdot \log_2[q_J(k)], \qquad \text{<12>}$$

where $q_J(k)$ is the pdf and $H_{J,angle}$ is angle entropy at resolution J.

4.3 Resolution Selection

The resolution selection for local angle follows the same way with resolution selection in computing global distance factor. At each resolution J, we compute the cost by:

$$f_{angle}(J) = \frac{H_{J,angle}}{\log_2 N} + \frac{e_{J,angle}}{e_{\max}} \qquad <13>$$

where $H_{J,angle}$ and $e_{J,angle}$ are angle entropy and quantization error at resolution J respectively, e_{max} is a constant (e_{max} =0.25).

Then we define the *local angle factor* of shape complexity as the minimum cost value over all resolutions:

$$C^e_{angle} = \min_j [f_{angle}(j)] \qquad <14>$$

4.4 Perceptual Smoothness

We now discuss a measure that explicitly encodes perceptual characteristics of shape. In computer vision it is well known that critical points in a 2D shape are those with large curvature [6]. Note that an entropy based measure (ref. eq. <14>) does not incorporate perception – it treats all angles equally. Thus, we define the perceptual smoothness of a shape using local angles:

$$P = \frac{1}{N}\sum_{i=1}^{N} \frac{e^{-\theta_i/\pi} - e^{-1}}{1-e^{-1}} \qquad <15>$$

Where P is perceptual smoothness, N is number of points of a shape and θ_i is local angle on the i[th] point. $(e^{-\theta i/\pi}-e^{-1})/(1-e^{-1})$ is the perceptual smoothness on i[th] point which reaches minimum value 0 when $\theta_i=\pi$ and reaches maximum value 1 when $\theta_i=0$. P is in the interval [0,1], the smaller value, the smoother the shape.

5. RANDOM TRACES

We now present an algorithm to compute the random factor of a shape using the distance between two traces through the point set.

5.1 Trace Searching Algorithm

The trace is a spanning tree of the set, where the edges are constructed using linear prediction. Since a spanning tree has N-1 edges for a graph with N vertices, our algorithm includes N-1 steps, in each of which we search for a new point and generate a new edge by following three sub-steps: (a) Search for the nearest neighbor v_{nn1} of a predicted point v_p obtained by using the previous point and current point v_c from non-passed points. If the distance between v_{nn1} and v_p is less than a threshold δ, connect v_{nn1} and v_c and goto v_{nn1}, otherwise goto sub-step b. (b) Search for the nearest neighbor v_{nn2} of current point v_c. If the distance between v_{nn2} and v_c is less than δ, connect v_{nn2} and v_c and goto v_{nn2}, otherwise goto sub-step c. (c) Search for a passed point v^* with minimum distance between itself to its nearest neighbor v_{nn3} from non-passed points, connect v^* and v_{nn3} and goto v_{nn3}.

We represent the spanning tree by an N*N matrix A. If the i[th] and j[th] points are connected in spanning tree, $A(i,j)=1$, otherwise $A(i,j)=0$. A trace for a truly random set will in general, depends upon the starting point. However, for regular shapes the difference between two traces starting from different points is very small – nearly independent of the starting point. This is because each point is highly correlated with its neighbors. For non-regular shapes, if two starting points are near each other, the difference between two traces can be small. But if two starting points are far away, the corresponding trace difference is large. Hence we select the two points with maximum distance as the starting points to reduce their correlations.

5.2 Trace Distance

In order to quantize the difference between two traces, a distance metric is needed to compute trace distance. We define the trace distance as the L[1] distance between two trace matrices:

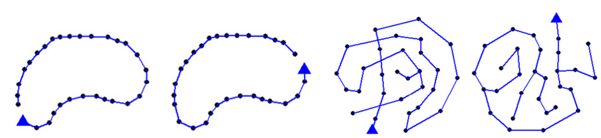
$$d(A_1, A_2) = \frac{1}{4\cdot(N-1)}\sum_{i=1}^{N}\sum_{j=1}^{N}\left|A_1(i,j) - A_2(i,j)\right|, \qquad <16>$$

Where A_1 and A_2 are two trace matrices, N is number of points.

Hence, the random factor of shape complexity is defined as the distance between two traces starting from two points with maximum distance:

$$R = d(A_{s_1}, A_{s_2}), \qquad <17>$$

where R is random factor, A_{s1} and A_{s2} are matrices of traces starting from two points s_1, s_2 with maximum distance respectively. In Figure 6, random traces of two shapes are shown. We can observe that the difference between two traces of regular shape and its corresponding random factor R is very small, whereas a random shape has large difference between two traces which results in large random factor.



(a) R=0.03 (b) R=0.62

Figure 6 Random Trace for. (a) a simple regular shape, (b) a random shape. The solid triangles are staring points.

6. THE SHAPE COMPLEXITY MEASURE

We now present our shape complexity measure that incorporates measures of global distance entropy (C^e_{dis}), local angle entropy (C^e_{angle}), perceptual smoothness (P) and the randomness measure (R). The global distance entropy, local angle entropy and perceptual smoothness have significant correlation to shape's structure. The random factor R tells us about stability of the structure. Random shapes can have widely different traces, while for non-random shapes, since each point is highly correlated with its neighbors, the trace is easy to find and stable. Therefore, we define the shape complexity in the form – *(1+R)*C(structural complexity)*.

In order to define the regularity based on C^e_{dis}, C^e_{angle} and P, a weight must be assigned to each of these parameters. Intuitively, if there are several descriptors to represent a shape, the simplest descriptor plays the most important role in determining shape simplicity. Thus, min(C^e_{dis}, C^e_{angle}) has larger weight than max(C^e_{dis}, C^e_{angle}). And the importance of P is between the min(C^e_{dis}, C^e_{angle}) and max(C^e_{dis}, C^e_{angle}).

Based on these considerations, we define the structural complexity of a shape as follows:

$$C = (1+R)\cdot(\alpha_1\cdot\min(C^e_{dist}, C^e_{angle}) + \alpha_2\cdot\max(C^e_{dist}, C^e_{angle}) + \alpha_3\cdot P) \qquad <18>$$
$$\alpha_1+\alpha_2+\alpha_3 = 1, \quad \alpha_1 \ge \alpha_3 \ge \alpha_2$$

where C is shape complexity, C^e_{dist} and C^e_{angle} are global distance factor and local angle factor respectively, R is random factor, P is perceptual smoothness, α_1, α_2 and α_3 are three constant weights between 0 and 1. In this paper we selected α_1=0.6, α_2=0.07, and

α_3=0.33. Theoretically, C is in the interval [0,2].Values close to 0 indicate a simple shape and values larger than 0.6 occur for very complex shapes. In practice, C is less than 1 for most of shapes.

7. EXPERIMENTS

In our experiments, there are two shape datasets – synthetic dataset and real world dataset. The synthetic dataset contains 97 shapes - 24 regular shapes such as circle, ellipse and simple polygons, 48 noisy shapes generated by adding noise into regular shapes and 25 hand-drawn shapes. Each shape has 30 points. The real dataset includes 22 dancer's shapes extracted by hand from dance video created by a world renowned choreographer Bill T. Jones. These shapes contain about 70 points (not fixed).

7.1 Results

Our experiments on the data indicate that the measure for complexity works well. Figure 7 shows complexities of six synthetic and six real world shapes in increasing order of their shape complexities (Eq.<18>). We can see that the circle has the smallest structural complexity, regular polygons have slightly greater complexities than the circle, polygon plus noise has larger complexity than regular polygons and the shapes that consist of random points have the most complexities.

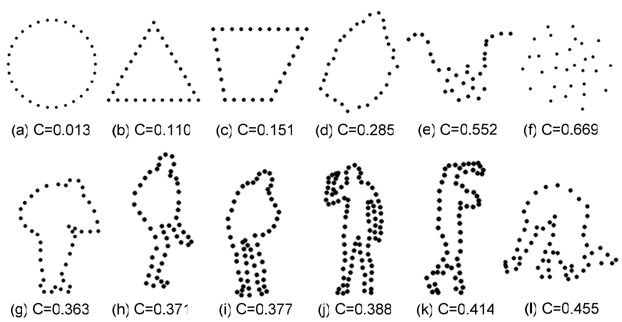


Figure 7 Examples of synthetic dataset. (a) circle, (b-c) regular polygons, (d) polygon plus noise, (e-f) random points. The second row shows results using real-world data.

7.2 User Studies

We conducted user studies to determine the relationship between our shape complexity measure and human perception. A group of 13 subjects with different backgrounds were presented with 100 pairs of shapes. The shapes were randomly selected from synthetic dataset and asked to determine if the first shape A (on the left) was more complex than the shape B on the right. To enable this, we presented the users with seven statements – "Shape A is much more complex than shape B", "more complex" etc. The user could only mark one statement to be true. Then we mapped an integer from -3 to 3 to each test pair as *perceptual complexity difference* (PCD) between two shapes. Positive integer means the first shape looks more complex than the second shape and vice versa. Large absolute values of the PCD imply greater perceptual complexity differences between the shapes.

The user studies indicate two clear results (a) our measure of complexity is correlated with human perception of complexity difference, (b) the sensitivity to the difference is proportional to complexity of the more complex shape – i.e. *user always compare the simpler shape with the more complex shape.* Intuitively this means that people can use small shape complexity differences to distinguish amongst simple shapes, and require large differences in complexity to tell apart more complex shapes.

We classified the 1300 shape pairs in the user studies into 7 classes based on the PCD assigned by users and computed the *expected structural complexity difference* (ESCD) for each class using Eq.<18> (shown in Figure 8 (a)). The figure shows that the complexity measure (using ECSD) is highly correlated with the perception of the difference. Figure 8 (b) shows sensitivity curve. The region above the curve is the differentiable region – i.e. with probability greater than 95%, for shapes whose complexity differences lie in that region are differentiable by the user. The difference threshold is small for low shape complexity.

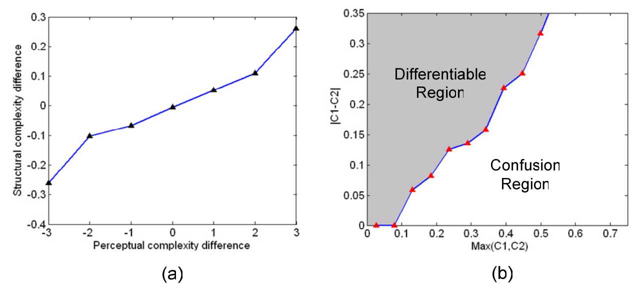


Figure 8 Relationship between perceptual complexity and structural complexity. (a) ESCD-PCD curve, (b) sensitivity curve showing the differentiable region (dark) and the region of confusion (light). Two shapes with shape complexity difference in this dark region can be differentiated with high probability.

8. CONCLUSION

In this paper, we have estimated the complexity of 2D shape by using a combination of three criteria – (a) entropy of the global distance distribution, (b) entropy of local angle distribution and (c) shape randomness. We evaluated our method by user studies on a synthetic and real-world dataset with excellent results. There is much room for improvement here – we plan to extend this work to 3D data, develop 3D shape distance measures that incorporate complexity of movement as well as human memory.

9. REFERENCES

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